Homework Exam 1 2025-2026

Model Solution

Deadline:

This homework exam has 1 question for a total of 9 points. You can earn an additional point by a careful preparation of your hand-in: using a good layout, good spelling, good figures, no sloppy notation, no statements like "The algorithm runs in $n \log n$." (forgetting the $O(..)$ and forgetting to say that it concerns time), etc. Use lemmas, theorems, and figures where appropriate.

Question 1 (9 points)

Let $r \in \mathbb{R}^2$ be a "red" point, and let B be a set of n "blue" points in $\mathbb{R}^2$. You can assume that the points are in general position; meaning that no two points have the same x-coordinate or the same y-coordinate, and that no three points lie on a line. A triangle is "bichromatic" when its vertices are either red or blue, and it has at least one vertex of either color. Develop an $O(n \log n)$ time algorithm to find a maximum area "bichromatic" triangle Δ^* on $\{r\}, B$.

Hint: You can use the following fact. A function $f[1..n] \to \mathbb{R}$ is *unimodal* if (and only if) it has a single (local) maximum. A maximum of f can be computed in $O(T \log n)$ time, where T is the time it takes to evaluate a single value $f(i)$ with $i \in [1..n]$. In particular, using the following function $\textsc{TernarySearch}([1, n], f)$:

function $\textsc{TernarySearch}([a..b], f)$
  $n \leftarrow b - a$
  if $n < 3$ **then** evaluate $f(i)$ for each $i \in [a..b]$ and return $\max_i f(i)$
  else
    $m_1 \leftarrow a + \lfloor n/3 \rfloor$; $m_2 \leftarrow a + \lfloor 2n/3 \rfloor$
    if $f(m_1) < f(m_2)$ **then** $\textsc{TernarySearch}([m_1.., b], f)$
    else$\textsc{TernarySearch}([a..m_2], f)$
    end if
  end if
end function

The key idea is in the following lemma, which proves that there exists an *optimal*, that is, maximum area, bichromatic triangle $\Delta^* = \Delta(r, b, b')$ for which b and b' are vertices of the convex hull $\mathcal{CH}(B)$ of B. This restricts the number of candidate triangles. Moreover, it actually gives us a way to find b' efficiently when we are given point b. This allows us to develop an $O(n \log n)$ time algorithm.

Lemma 1. *Let $b \in B$ be a point that does not appear on $\mathcal{CH}(B)$, and let H^+ be any halfplane whose bounding line goes through b. Then H^+ contains a point from B.*

Proof. Assume, by contradiction, that b does not appear on $\mathcal{CH}(B)$, yet H^+ is empty. Since b lies in the interior of $\mathcal{CH}(B)$, it follows $\mathcal{CH}(B)$ must intersect H^+ in some region R that has non-zero area. Now consider the set $\mathcal{CH}(B) \cap H^-$ (where $H^- = \mathbb{R}^2 \setminus H^+$ is the other halfplane defined by the bounding line of H^+). Since both $\mathcal{CH}(B)$ and H^- are convex, so is $\mathcal{CH}(B) \cap H^-$. Furthermore, since H^+ contains no points from B, we have that $B \subseteq \mathcal{CH}(B) \cap H^-$. Since R is has non-zero area, and thus $\mathcal{CH}(B) \cap H^- \subset \mathcal{CH}(B)$. This contradicts the definition of $\mathcal{CH}(B)$. This completes the proof. □

Lemma 2. *Let r and b be fixed, and let b' be a vertex of $\mathcal{CH}(B)$ whose distance to the line ℓ_{rb} through r and b is maximal. The triangle $\Delta(r, b, b')$ has maximum area among all bichromatic triangles with vertices r and b.*

Proof. Assume, by contradiction, that $\Delta(r, b, b')$ is a maximal area triangle, but that b' is not a vertex of $\mathcal{CH}(B)$ for which the distance to the line ℓ_{rb} is maximal, nor does there exist a triangle $\Delta(r, b, b'')$ with larger area.

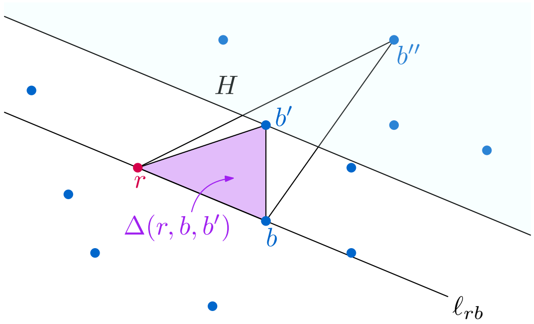

Figure 1: If H is non-empty, triangle $\Delta(r, b, b')$ cannot be a maximal area triangle.

Now consider the half-plane H whose bounding line is parallel to ℓ_{rb} and goes through point b', and such that $b \notin H$. See Fig. 1. Since (by assumption) b' is not a vertex of the convex hull, Lemma 1 gives us that H cannot be empty. So, let $b'' \in B$ be a point in H.

Observe that the distance from a point $q \in \mathbb{R}^2$ to line ℓ_{rb} is the height of $\Delta(r, b, q)$. Since the distance from b'' to ℓ_{rb} is larger than the distance from b' to ℓ_{rb}, it thus follows that the height of $\Delta(r, b, b'')$ is larger than that of $\Delta(r, b, b')$. Since both triangles have the same base, the area of $\Delta(r, b, b'')$ is also larger than that of $\Delta(r, b, b')$. Contradiction. □

Lemma 3. *There exists an optimal triangle $\Delta^* = \Delta(r, b, b')$ for which (i) b is a vertex of $\mathcal{CH}(B)$, and (ii) b' is a vertex of $\mathcal{CH}(B)$ whose distance to the line ℓ_{rb} through r and b is maximal.*

Proof. By applying Lemma 2 twice: We first fix r and b', and use the Lemma 2 to obtain that b must be a vertex of $\mathcal{CH}(B)$, thus establishing (i). We then apply Lemma 2 once more fixing points r and b to obtain (ii). □

Fix a point $b \in \mathcal{CH}(B)$, and let H^+ an H^- denote the two halfspaces bounded by the line rb. Let $b = b_1, .., b_k$ denote the points on $\mathcal{CH}(B)$ in order along $\mathcal{CH}(B)$ in H^+, and let $h(i)$ denote the distance from b_i to the line ℓ_{rb}.

Lemma 4. *The function h is unimodal.*

Proof. For ease of description, rotate and translate the plane so that H^+ is bounded from below by the x-axis (so r and b lie on the x-axis). This way, $h(i)$ is simply the y-coordinate of b_i. We now argue that h is unimodal.

Assume, by contradiction, that h has two local maxima at i and j, with $i < j - 1$. Let $m \in \{i + 1, j - 1\}$ minimize h (among $i + 1$ and $j - 1$). Hence, the y-coordinate of b_m is smaller than that of b_i and b_j. It then follows b_m lies strictly below the oriented line through b_i and b_j. However, therefore b_m cannot lie on $\mathcal{CH}(B)$ (or at least on the portion of $\mathcal{CH}(B)$ in between b_i and b_j). Contradiction. □

Analogous to Lemma 4 we the distance from the vertices of $\mathcal{CH}(B) \cap H^-$ is unimodal. This, together with Lemma 2 then suggests an $O(\log n)$ time algorithm to find a triangle $\Delta(r, b, b')$ with maximal area among all bichromatic triangles with vertices r and b (provided we have access to $\mathcal{CH}(B)$):

- Using a binary search, we find the last vertex b_k such that $b = b_1, .., b_k$ lie in H^+.
- Using the algorithm $\textsc{TernarySearch}([1..k], h)$, we then find a vertex $b_i \in b_1, .., b_k$ with maximum distance to ℓ_{rb}. The triangle $\Delta(r, b, b_i)$ has maximum area among $b_1, .., b_k$.
- We repeat the previous step for the other points on $\mathcal{CH}(B)$ (that are in H^-).

It now follows that we have an $O(n \log n)$ time algorithm to compute an maximal area bichromatic triangle. We first compute the convex hull $\mathcal{CH}(B)$, and then use the above approach for each vertex $b \in \mathcal{CH}(B)$. We report the triangle $\Delta(r, b, b')$ with maximal area that we find. Correctness follows from Lemmas 3 and 4.

Computing the convex hull takes $O(n \log n)$ time. The above approach takes $O(\log n)$ time per candidate point b, and there are $O(n)$ candidate points b. We thus obtain the following result.

Theorem 5. *Given a point $r \in \mathbb{R}^2$, and a set B of n points in $\mathbb{R}^2$, we can compute a maximal area bichromatic triangle $\Delta(r, b, b')$ on r, B in $O(n \log n)$ time.*