# Final Exam 2024-2025

29 January 2025, 17:00-19:30

This exam has 1 questions for a total of 10 points. You can earn an additional 10 points if you write readable, unambiguous, and technically correct. No statements like "The algorithm runs in $n \log n$." (forgetting the $O(..)$ and forgetting to say that it concerns time), etc. Your final grade will be the number of points divided by 10.

Read every question carefully (!), make sure you understand it, and be sure to answer the question. Answer questions in sufficient but not too much detail. You may **not** use the textbook, or any other notes during the exam. Be sure to put your name on every piece of paper you hand in. Good Luck!

**Question 1 (10 points)**

For each of the following tasks, state the running time for the best possible algorithm to perform the task. If the algorithm is deterministic, give the worst case running time. If the algorithm is randomized, indicate this and give the expected running time. Use $k$ to denote the output size if applicable.
Stating only the running time is sufficient, no need to explain your answers in detail.

(a) Given a set $P$ of $n$ points in $\mathbb{R}^3$, constructing a range tree on $P$.
(b) Given a set $\{q\} \cup P$ of $n$ points in $\mathbb{R}^2$, computing the smallest disk that contains all points in $P$ and has $q$ on its boundary.
(c) Given a Delaunay triangulation of a set $P$ of $n$ points in $\mathbb{R}^2$; computing the Euclidean minimum spanning tree of $P$.
(d) Given a segment tree storing a set of $n$ horizontal line segments in $\mathbb{R}^2$, in particular, in which every canonical subset is stored in an array, ordered from bottom to top, querying the segment tree with a point $q$ to report all segments below $q$.
(e) Computing the arrangement of a set of $n$ lines in $\mathbb{R}^2$.

**Question 2**

Let $P$ and $Q$ be two simple polygons, with $n$ and $m$ vertices, respectively. Our goal is to compute $P \setminus Q$; i.e. the difference of $P$ "minus" $Q$.

(a) *(10 points)* How can we efficiently/compactly represent the result $P \setminus Q$? Briefly explain what the worst case complexity of $P \setminus Q$ is. Give both an upper and a lower bound.
(b) *(5 points)* Briefly (at most one paragraph) describe how we can efficiently compute $P \setminus Q$ using an output sensitive algorithm.
(c) *(5 points)* Give the running time of your algorithm. Briefly explain your answer.
(d) *(7 points)* Suppose $P$ and $Q$ are convex polygons instead. Does this affect the worst case running time of your algorithm? Argue why, or why not.

**Question 3**

Let $\ell, m$ and $n$ be three lines, ordered by slope, that all intersect in a point $s$, and let $\overline{pq}$ be a line segment with endpoints $p$ and $q$ whose supporting line passes below $s$. Furthermore, $\overline{pq}$ intersects only line $m$.

(a) *(6 points)* Draw the configuration described above in both the primal and in the dual plane. Clearly label the objects $p, q, m, n, \ell, s$, and $\overline{pq}$ and their duals.
(b) *(6 points)* Translate the above paragraph to its dual form. It does not have to be a direct translation, but make sure to capture the geometric relations that are described.

**Question 4**

Let $H$ be a set of $n$ planes in $\mathbb{R}^3$, and let $h$ be plane (not necessarily appearing in $H$).
(a) *(10 points)* Argue that the total complexity of the arrangement of $H$ is $O(n^3)$.
(b) *(3 points)* Define/explain what the zone of $h$ with respect to $H$ is.

**Question 5 (10 points)**

Let $P$ be a set of $n$ points in $\mathbb{R}^2$. Prove that any edge in the Euclidean minimum spanning tree on $P$ is also an edge in the Delaunay triangulation of $P$.

**Question 6**

Let $P$ be a set of $n$ points in $\mathbb{R}^2$. You may assume all points in $P$ have unique $x$ and $y$-coordinates. Consider the following recursive algorithm to compute the lower hull of $P$, represented as a chain of its vertices in left to right order.

1. If $P$ contains just one point $p$; return the singleton chain $p$.
2. Otherwise, compute the median $x$-coordinate $m$ among the points in $P$.
3. Compute the edge $\overline{\ell r}$ of the lower hull $LH(P)$ intersected by the vertical line at $x$-coordinate $m$ by calling a subroutine **Magic**$(P, m)$. (For ease of description; assume that this edge is unique).
4. Partition the point set $P$ into three sets: $P_{\leq}$ containing the points whose $x$-coordinate is at most $\ell_x$, $P_{\geq}$ whose $x$-coordinate is at least $r_x$, and the remaining subset $P_m = P \setminus (P_{\leq} \cup P_{\geq})$.
5. Recursively compute the lower hulls $LH(P_{\leq})$ and $LH(P_{\geq})$.
6. Concatenate the chains $LH(P_{\leq})$ and $LH(P_{\geq})$, and return the resulting chain.

(a) *(10 points)* Analyze the running time of the above algorithm, assuming that the subroutine **Magic**$(P, m)$ takes $O(n \log n)$ time when called on a set of size $n$.
Your analysis should be complete and self contained. In particular, you may not appeal to the master theorem.
(b) *(8 points)* Is there a more efficient implementation of the algorithm **Magic**$(P, m)$? Explain your answer in one paragraph.